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Complementary Forms of the Variational Principle for Heat Conduction and Convection*

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ABSTRACT: *Complementary forms of the variational principle for heat convection are developed. They are applicable to non-homogeneous fluids with temperature dependent properties and include the case of turbulent flow. For linear problems following a general procedure introduced by the author for non-selfadjoint operators, the variational principle is expressed in operational symbolism which includes implicitly a convolution form.*

Introduction

The fluid may be heterogeneous with temperature dependent properties different for each fluid particle. Turbulent flow is included. The basic equations for heat convection are (1)

$$J_i = -k_{ij}\frac{\partial \theta}{\partial x_j} \tag{1}$$

$$\frac{Dh}{Dt} + \frac{\partial J_i}{\partial x_i} = 0. \tag{2}$$

The following quantities and definitions are used, $D/Dt = \partial/\partial t + v_i(\partial/\partial x_i)$. It is assumed that the velocity v_i satisfies at least approximately the condition $\partial v_i/\partial x_i = 0$ of incompressibility, x_i = coordinates of a fluid particle at time t. θ = temperature field. $k_{ij} = k_{ji} = k_{ij}(x_k, t, \theta)$, sum of the molecular and "turbulent conductivity". X_i = coordinates of a fluid particle at $t = 0$ with $x_i = x_i(X_k, t)$ assumed to be given functions of initial coordinates and time

$$h = \int_0^\theta c(X_k, \theta)\, d\theta,$$

heat content of a fluid particle where $c(X_k, \theta)$ is the heat capacity of a fluid particle per unit volume. The particular case of pure conduction is included in these equations by putting $v_i = 0$.

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Fundamental Variational Form

Introduction of the heat displacement as a variable conjugate to the temperature extended to heat transfer the fundamental methods of analytical dynamics. This was developed in detail earlier. The most general variational formulation in terms of heat displacement was derived in a recent paper (1) and is expressed as follows.

Equation 2 which expresses heat conservation is satisfied identically by putting

$$J_i = \frac{\partial H_i}{\partial t} - v_i h \tag{3}$$

$$h = \frac{\partial H_i}{\partial x_i}. \tag{4}$$

The vector H_i is an unknown heat displacement field. Equation 1 is then replaced by the variational equivalent

$$\iiint_\tau \left(\frac{\partial \theta}{\partial x_i} + \lambda_{ij} J_j\right) \delta H_i \, d\tau = 0. \tag{5}$$

The λ_{ij}'s are the elements of the inverse matrix of k_{ij}

$$[\lambda_{ij}] = [k_{ij}]^{-1}. \tag{6}$$

The heat displacement field is expressed as

$$H_i = H_i(q_j, x_k, t) \tag{7}$$

by means of n generalized coordinates q_j. We have shown that the variational Eq. 5 implies n differential equations for q_j in the Lagrangian form

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \tag{8}$$

where

$$\dot{q}_i = \frac{dq_i}{dt}$$

$$\left.\begin{aligned} V &= \iiint_\tau d\tau \int_0^\theta \theta \, dh = \text{thermal potential} \\ D &= \frac{1}{2}\iiint_\tau \lambda_{ij} J_i J_j \, d\tau = \text{dissipation function} \\ Q_i &= -\iint_A \theta n_j \frac{\partial H}{\partial q_i} dA \text{ thermal force} \end{aligned}\right\} \tag{9}$$

The surface integral defining the thermal force is extended to the boundary A of the volume τ and n_j is the unit outward normal to the surface.

Complementary Variational Form

An obvious alternate procedure is to satisfy identically Eq. 1 which is then considered as defining J_i in terms of an unknown temperature field θ. The energy conservation Eq. 2 is now replaced by its equivalent variational form

$$\iiint_\tau \left(\frac{Dh}{Dt} + \frac{\partial J_i}{\partial x_i}\right) \delta\theta \, d\tau = 0. \tag{10}$$

The convected derivative term may be written

$$\frac{Dh}{Dt} = c\frac{\partial \theta}{\partial t} + cv_i \frac{\partial \theta}{\partial x_i}. \tag{11}$$

The fluid may be heterogeneous, with temperature-dependent properties. Hence, in general, $c = c(x_k, t, \theta)$. The temperature field is made equal to a function of n generalized coordinates q_j

$$\theta = \theta(q_j, x_k, t). \tag{12}$$

Hence,

$$\delta\theta = \frac{\partial \theta}{\partial q_i}\delta q_i \tag{13}$$

$$\dot{\theta} = \frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial q_i}\dot{q}_i + \left[\frac{\partial \theta}{\partial t}\right]. \tag{14}$$

The term $[\partial\theta/\partial t]$ is put in brackets to indicate that it is different from $\partial\theta/\partial t$ in Eq. 11. From Eq. 14 we derive

$$\frac{\partial \theta}{\partial q_i} = \frac{\partial \dot{\theta}}{\partial \dot{q}_i} \tag{15}$$

hence

$$\delta\theta = \frac{\partial \dot{\theta}}{\partial \dot{q}_i}\delta q_i. \tag{16}$$

After integrating by parts the term $\partial J_i/\partial x_i$ in Eq. 10 and using for $\delta\theta$ the values Eqs. 13 and 16 the variational Eq. 10 becomes

$$\iiint_\tau \left(c\dot{\theta}\frac{\partial \dot{\theta}}{\partial \dot{q}} + cv_j \frac{\partial \theta}{\partial x_j}\frac{\partial \theta}{\partial q_i} - J_j \frac{\partial^2 \theta}{\partial x_j \partial q_i}\right)\delta q_i \, d\tau + \iint_A J_j n_j \frac{\partial \theta}{\partial q_i}\delta q_i \, dA = 0. \tag{17}$$

We derive the following n differential equations for the generalized coordinates

$$\frac{\partial G}{\partial \dot{q}_i} + L_i + K_i = P_i \tag{18}$$

with

$$
\begin{aligned}
G &= \frac{1}{2}\iiint_{\tau} c\theta^2 \, d\tau \\
L_i &= \iiint_{\tau} cv_j \frac{\partial\theta}{\partial x_j}\frac{\partial\theta}{\partial q_i}\, d\tau \\
K_i &= -\iiint_{\tau} J_j \frac{\partial^2\theta}{\partial x_j \partial q_i}\, d\tau \\
P_i &= -\iint_{A} J_j n_j \frac{\partial\theta}{\partial q_i}\, dA.
\end{aligned}
\tag{19}
$$

The term L_i is convective, while K_i is dissipative. In principle, J_j is expressed in terms of θ by Eq. 1. However, the assumed distribution of θ may imply discontinuous derivatives. In this case accuracy will be improved by using a smooth distribution of J_i which departs from the strict definition Eq. 1. The boundary term P_i disappears at boundaries where there are no heat fluxes ($J_i = 0$) or at those where the temperatures are given functions of the time.

The term P_i appears: a) at boundaries where the heat flux $J_i n_i$ is a prescribed function of t and x_k; b) at boundaries with heat transfer and unknown temperatures. Note that Eqs. 18 are applicable for the general case of a nonhomogeneous fluid with temperature dependent properties; hence, for c and k_{ij}, functions of x_k, t, and θ with or without turbulent flow.

Some particular cases of interest are discussed next.

Temperature-independent Conductivity

If the thermal conductivity $k_{ij} = k_{ji}$ is independent of the temperature the term K_i may be written

$$K_i = \frac{\partial D}{\partial q_i} \tag{20}$$

with a dissipation function

$$D = \frac{1}{2}\iiint_{\tau} k_{ij} \frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}\, d\tau. \tag{21}$$

Equations 18 become

$$\frac{\partial G}{\partial \dot{q}_i} + \frac{\partial D}{\partial q_i} + L_i = P_i. \tag{22}$$

This result may be extended to a temperature-dependent conductivity provided it is of the form

$$k_{ij} = k_{ij}'(x_k, t) f(\theta). \tag{23}$$

In this case we put

$$u = \int_0^\theta f(\theta)\, d\theta \tag{24}$$

and write

$$J_i = -k_{ij}' \frac{\partial u}{\partial x_j}$$

$$h = \int_0^\theta c'(x_k, \theta)\, d\theta \tag{25}$$

$$c' = \frac{c}{f(\theta)}.$$

The variable u plays the role of a temperature in a medium of heat capacity c' and thermal conductivity k_{ij}'. The latter is now independent of the temperature.

Heat Conduction. For pure heat conduction without convection ($v_i = 0$), the term L_i disappears. Equation 18 is then reduced to

$$\frac{\partial G}{\partial \dot{q}_i} + K_i = P_i \tag{26}$$

and Eq. 22 for temperature-independent conductivity becomes

$$\frac{\partial G}{\partial \dot{q}_i} + \frac{\partial D}{\partial q_i} = P_i. \tag{27}$$

Operational Method. Assuming that c and k_{ij} depend only on the coordinates and combining Eqs. 1 and 2 leads to the linear operational equation

$$pc\theta + cv_i \frac{\partial \theta}{\partial x_i} - \frac{\partial}{\partial x_i}\left(k_{ij} \frac{\partial \theta}{\partial x_j}\right) = 0 \tag{28}$$

with the operator

$$p = \frac{\partial}{\partial t}. \tag{29}$$

Variational principles in operational form for dissipative systems and nonselfadjoint equations were introduced by the author (2, 3) in the context of coupled thermoelasticity and viscoelasticity, and applied extensively to many types of problems. The method is based on a fundamental property of the operational formalism, namely, that the result may be interpreted either as an integro-differential expression or as a Laplace transform. For example, the differential Eq. 28 may be considered to apply to the Laplace transform of θ. We then write

Eq. 28 in the form

$$\iiint_\tau \left[pc\theta + cv_i \frac{\partial \theta}{\partial x_i} - \frac{\partial}{\partial x_i}\left(k_{ij} \frac{\partial \theta}{\partial x_j}\right)\right] \delta\theta \, d\tau = 0. \tag{30}$$

The symbolic significance of this equation is obtained by considering θ to represent the Laplace transform of itself, namely

$$\mathcal{L}(\theta) = \int_0^\infty e^{-pt'}\theta(t')\, dt'. \tag{31}$$

As a consequence, Eq. 30 leads to the following operator-variational principle,

$$p\delta V + \delta D + \iiint_\tau cv_i \frac{\partial \theta}{\partial x_i} \delta\theta \, d\tau = -\iint_A J_i n_i \, \delta\theta \, dA \tag{32}$$

where

$$V = \frac{1}{2} \iiint_\tau c\theta^2 \, d\tau. \tag{33}$$

For pure heat conduction, Eq. 32 is simplified to

$$p\delta V + \delta D = -\iint_A J_i n_i \, \delta\theta \, dA. \tag{34}$$

This is the complementary form of the operator-variational principle derived earlier in the context of coupled thermoelasticity (3).

Various Interpretations of the Operator-Variational Principle

It is possible to interpret the variational Eq. 32 in various ways.

Consider θ to be represented as a linear superposition of fixed configurations

$$\theta = \theta_i(x) q_i(t) \tag{35}$$

where $\theta_i(x)$ are given scalar fields and q_i are generalized coordinates. Equation 35 remains formally the same when θ and q_i represent symbolically their Laplace transforms. With this interpretation in mind we substitute expression 35 into Eq. 32. We obtain

$$p \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial q_i} + L_i = P_i. \tag{36}$$

Going back to time variables with $p = d/dt$ relations 36 represent a complete set of differential equations for the generalized coordinates q_i.

The operator-variational principle Eq. 32 may also be interpreted in terms of convolutions since the various terms are products of Laplace transforms. Consider for example the term

$$p\delta V = \frac{1}{2}\iiint_{\tau} cp\theta\,\delta\theta\,d\tau. \tag{37}$$

By a well known property concerning the product of Laplace transforms

$$p\theta\,\delta\theta = \int_0^t \dot{\theta}(t - t')\,\delta\theta(t')\,dt'. \tag{38}$$

Similar convolutions are obtained for the other terms.

As can be seen the operational symbolism provides a powerful and compact formulation of variational principles which may be interpreted immediately in terms of differential and integral equations or convolutions. Among many advantages the operational symbolism brings out the important property of commutativity.

The method is quite general and is applicable to time operators which are not equired to be self-adjoint.

Comparative Accuracy of the Complementary Form

Attention is called to a particular feature of the complementary form of the variational principle, namely, that it involves a space differentiation of the temperature field. As a consequence, it will generally be less accurate than the fundamental form in numerical applications.

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