Short Notes

M. A. BIOT

FURTHER DEVELOPMENT OF THE THEORY OF INTERNAL BUCKLING OF MULTILAYERS

Abstract: The approximate theory of internal buckling of multilayers is obtained by a direct method and generalized to nonsinusoidal deformations. The numerical discussion includes the case of unequal thickness of competent and incompetent layers. Simple expressions are derived for the dominant wave length for rigid confinement or self-confinement showing the influence of interstitial flow. The theory is presented in the context of viscous media using both solid and fluid mechanics and is valid for large compressive strain and moderate slopes. Its applicability to elastic and viscoelastic media is indicated.

Introduction

For geological applications it is important to supplement exact theories of buckling of multilayered structures by simplified approximate treatments which bring out the essential mechanics and lead to simple formulas. For internal buckling of confined multilayers, such an approach was initiated in two previous papers (Biot, 1963; 1964a). In this paper the equations that govern internal buckling are derived in a different and more direct way. They are also expressed in more general form, as a set of field equations with two unknowns: the vertical deflection and the vertical stress. These field equations are not restricted to sinusoidal deflections.

Values of the dominant wave length were given earlier (Biot, 1964a) for competent and incompetent layers of equal thickness. In this paper we have extended the numerical discusssion to layers of unequal thickness and obtained results quite similar to those of the previous analysis, showing that the thickness of the competent layer, as well as the confinement thickness, is a significant parameter.

Internal buckling may be caused by rigid confinement or by self-confinement when occurring in a medium of infinite extent. Either case follows from the same analysis. The theory is presented in the context of viscous media and is derived in two different ways: from solid and fluid mechanics. Both methods lead to equations which for all practical purpose are identical. The results are valid for large compressive strain with variable thickness of the layers.

As will be pointed out further on, equations applicable to elastic and viscoelastic media are immediately obtained by viscoelastic correspondence.

Equilibrium Equations For Plate Buckling

Consider a plate of thickness $2h$ under an average compressive stress P along its axis (Fig. 1). We will assume plane strain in the x,y plane of the figure. When the plate is deformed with moderate slopes, a compressive stress approximately equal to P continues to act along the deformed axis (Fig. 1). At the same time the deformation generates a bending moment $\mathfrak{M}$ and a total shear $\mathfrak{N}$ acting over a cross section. This cross section is assumed to be of unit width in the direction perpendicular to the x,y plane. Hence $\mathfrak{M}$ and $\mathfrak{N}$ have the dimensions of a moment and force per unit width. Also per unit width and per unit distance along the axis we apply a vertical load q and a clockwise moment m. The equilibrium conditions of these forces for the deformed plate with moderate slopes are approximately

$$\frac{\partial \mathfrak{M}}{\partial x} + m = \mathfrak{N}$$

$$\frac{\partial \mathfrak{N}}{\partial x} + q = 2Ph\frac{\partial^2 v}{\partial x^2}. \qquad (1)$$

The vertical deflection is denoted by v. Except for the additional external moment m, these

Geological Society of America Bulletin, v. 76, p. 833–840, 5 figs., July 1965

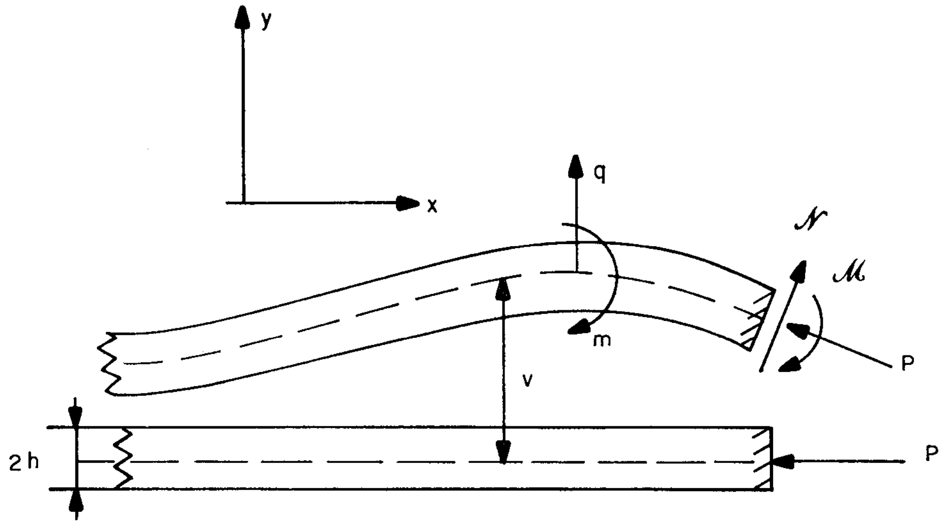

Figure 1. Forces and stresses on a composite plate containing one competent layer

equations are the same as those considered in the previous theory of similar folding (Biot, 1965a). Elimination of $\mathfrak{M}$ yields

$$\frac{\partial^2 \mathfrak{M}}{\partial x^2} = -q - \frac{\partial m}{\partial x} + 2Ph \frac{\partial^2 v}{\partial x^2}. \qquad (2)$$

A derivation of this equation is also given in an earlier paper dealing with the effect of interfacial adherence on viscous and viscoelastic folding (Biot, 1959). It can be shown that equation (2) is a consequence of the general mechanics of initially stressed media (Biot, 1965b, p. 127).

Buckling Equation for a Competent Layer in a Multilayered Structure

Consider a multilayered structure of viscous, incompressible material. Competent layers of thickness h_1 alternate with incompetent layers of thickness h_2 (Fig. 2). The system may be regarded as a stacking of plates, each plate being composed of a competent layer sandwiched between two incompetent layers of thickness $h_2/2$. The total thickness of the composite plate is denoted by $2h = h_1 + h_2$. We shall apply equation (2) to this composite plate. We

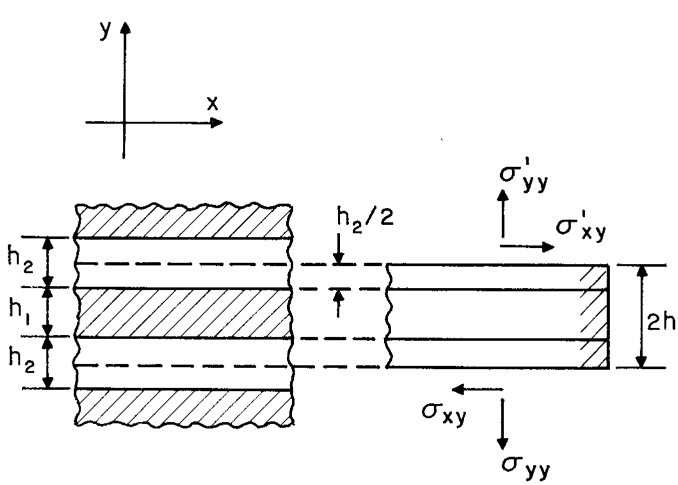

Figure 2. Structure of composite plate containing one competent layer

assume that the bending moment is generated entirely by the competent layer of viscosity η_1. This bending moment is[1]

$$\mathfrak{M} = -\tfrac{1}{3}\eta_1 h_1^3 \frac{\partial^2 \dot{v}}{\partial x^2}, \tag{3}$$

where $\dot{v}$ is the time derivative of the vertical deflection of the axis.

Tangential stresses σ_{xy}' and σ_{xy} are also acting on top and bottom of the composite plate. They generate an external moment

$$m = h(\sigma_{xy}' + \sigma_{xy}). \tag{4}$$

The tangential stress may be written approximately

$$\sigma_{xy}' \cong \sigma_{xy} \cong \eta_t \frac{\partial \dot{v}}{\partial x}. \tag{5}$$

Hence,

$$m = 2h\eta_t \frac{\partial \dot{v}}{\partial x}. \tag{6}$$

The average tangential viscosity coefficient of the multilayer is (Biot, 1963; 1964a)

$$\eta_t = 1/(\alpha_1/\eta_1 + \alpha_2/\eta_2), \tag{7}$$

where η_2 is the viscosity of the incompetent layer, whereas $\alpha_1 = h_1/(2h)$ and $\alpha_2 = h_2/(2h)$ represent the fractions of the total thickness occupied respectively by the competent and incompetent layers.

The vertical force q is the difference between the vertical stresses σ_{yy}' and σ_{yy} at the top and bottom of the composite plate (Fig. 1). Hence,

$$q = \sigma_{yy}' - \sigma_{yy} \cong 2h \frac{\partial \sigma_{yy}}{\partial y}. \tag{8}$$

In this expression the vertical stress σ_{yy} is assumed to be distributed continuously along the vertical axis y. With the values obtained from equations (3), (6), and (8), equation (2) becomes

$$\tfrac{1}{3}\eta_1 h_1^3 \frac{\partial^4 \dot{v}}{\partial x^4} + 2Ph\frac{\partial^2 v}{\partial x^2} - 2h\eta_t \frac{\partial^2 \dot{v}}{\partial x^2} - 2h\frac{\partial \sigma_{yy}}{\partial y} = 0. \tag{9}$$

This equation contains two unknowns, v and σ_{yy}. Another equation will now be derived by considering the average vertical compressibility of the multilayered structure.

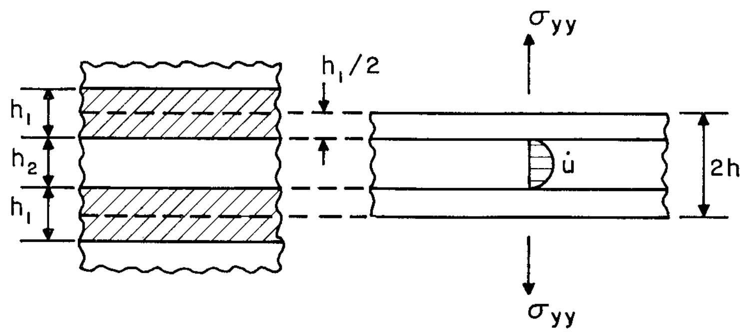

Figure 3. Interstitial flow in the incompetent layer

Vertical Compressibility and Interstitial Flow

Again we regard the multilayered structure as a stacking of composite plates. However, this time the composite plate is made up of an incompetent layer sandwiched between two competent layers, each of thickness $h_1/2$ (Fig. 3). Under a vertical stress σ_{yy} this composite plate will exhibit a vertical strain rate. If the strain is assumed to be the same in the two materials the rate of change of the total thickness is

$$\dot{v}' - \dot{v} = \frac{h}{2\eta_n}\sigma_{yy}. \tag{10}$$

In this expression $\dot{v}'$ and $\dot{v}$ are the time derivatives of the vertical displacements on top and bottom of the composite plate, and η_n is the

[1] This moment acts only in the competent layer of thickness h_1. A derivation of equation (3) was given previously (Biot, 1961).

average vertical viscosity coefficient of the multilayered structure (Biot, 1963; 1964a),

$$\eta_n = \alpha_1\eta_1 + \alpha_2\eta_2 . \tag{11}$$

Note that v represents the vertical displacement of the center line of a competent layer.

As already pointed out in a previous analysis (Biot, 1964a) an additional vertical strain may become important owing to the occurrence of interstitial flow in the incompetent layer. This flow is represented by a horizontal velocity distribution $\dot{u}$, approximately parabolic across the thickness of the incompetent layer (Fig. 3). We write this velocity distribution

$$\dot{u} = \left(1 - \frac{4y^2}{h_2^2}\right) f(x) . \tag{12}$$

In this expression we have assumed the x axis to coincide with the center line of the incompetent layer. Since the layer is incompetent the two normal stress components, σ_{xx} and σ_{yy}, inside the layer are approximately equal. Hence we put $\sigma_{xx} = \sigma_{yy}$. With this assumption the equilibirum condition for the stresses becomes

$$\frac{\partial \sigma_{yy}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 . \tag{13}$$

The shear stress σ_{xy} in the layer is

$$\sigma_{xy} = \eta_2 \frac{\partial \dot{u}}{\partial y} . \tag{14}$$

Combining equations (12), (13), and (14) we find

$$f(x) = \frac{h_2^2}{8\eta_2} \frac{\partial \sigma_{yy}}{\partial x} . \tag{15}$$

Because of incompressibility the local rate of increase of thickness of the incompetent layer due to interstitial flow is

$$\dot{h}_2 = -\int_{-\frac{1}{2}h_2}^{\frac{1}{2}h_2} \frac{\partial \dot{u}}{\partial x} dy = -\tfrac{2}{3} h_2 \frac{df}{dx} . \tag{16}$$

With the value (15) for $f(x)$ this expression becomes

$$\dot{h}_2 = -\frac{h_2^3}{12\eta_2} \frac{\partial^2 \sigma_{yy}}{\partial x^2} . \tag{17}$$

When interstitial flow is taken into account the total thickening of the composite plate is obtained by adding $\dot{h}_2$ to expression (10); hence,

$$\dot{v}' - \dot{v} = \frac{h}{2\eta_n} \sigma_{yy} - \frac{h^3{}_2}{12\eta_2} \frac{\partial^2 \sigma_{yy}}{\partial x^2} . \tag{18}$$

Finally we assume an averaged smooth distribution of $\dot{v}$ in the vertical direction. We put $\dot{v}' - \dot{v} = 2h(\partial \dot{v}/\partial y)$ and write equation (18) as

$$2h \frac{\partial \dot{v}}{\partial y} = \frac{h}{2\eta_n} \sigma_{yy} - \frac{h_2^3}{12\eta_2} \frac{\partial^2 \sigma_{yy}}{\partial x^2} . \tag{19}$$

Thus we have derived two simultaneous equations, (9) and (19), for the two unknowns, v and σ_{yy}.

Internal Buckling under Self-Confinement

Let us assume a sinusoidal distribution of v and σ_{yy} along x and y. We put

$$\begin{aligned} v &= V \cos \xi l y \cos l x \\ \sigma_{yy} &= -S \sin \xi l y \cos l x . \end{aligned} \tag{20}$$

Substituting these values in equations (9) and (19) and eliminating S yields

$$P \frac{V}{\dot{V}} = \eta_t + \left(\frac{\pi}{n}\right)^2 \eta_n \frac{\delta}{\gamma^2} + \frac{16}{3} \alpha_1^3 \eta_1 \gamma^2 . \tag{21}$$

We have put

$$\begin{aligned} \gamma &= \tfrac{1}{2} l h \\ \delta &= \frac{1}{1 + \kappa\gamma^2} \\ \kappa &= \frac{16}{3} \frac{\eta_n}{\eta_2} \alpha_2^3 \\ \xi &= \frac{\pi}{2n\gamma} . \end{aligned} \tag{22}$$

The quantity n represents the number of layers contained in the half vertical wave length H. This can be seen by writing the value of H as

$$H = \frac{\pi}{\xi l} = nh . \tag{23}$$

The wave length $\mathfrak{L}$ along the layers is related to γ and l by the relation

$$\mathfrak{L} = \frac{2\pi}{l} = \frac{\pi h}{\gamma} . \tag{24}$$

The pattern of self-confinement folding represented by equations (20) is shown schematically in Figure 4a and is the same as previously derived for anisotropic elastic media and multilayered structures (Biot, 1963; 1964a). The term *self-confinement* was introduced to indicate that this type of folding occurs in a medium of infinite extent and does not require the presence of any confining boundaries.

Under the average compressive stress P the structure undergoes a uniform horizontal shortening of the layers and a corresponding thickening in the vertical direction. Folding patterns as shown in Figure 4a will develop spontaneously with rates of growth, depending on the vertical and horizontal wave lengths. They are triggered by the particular distribution of irregularities initially present in the structure. The time history of the pattern can be evaluated by expanding the initial disturbance in a double Fourier series and proceeding exactly as in the similar problem for the case of a single layer (Biot and others, 1961).

Actually during the process of folding, the wave length $\mathcal{L}$ decreases due to the over-all compressive strain. Correspondingly the wave length $2H$ in the vertical direction increases. This may be taken into account by noting that

$$\gamma = \frac{\pi h}{\mathcal{L}} = \frac{\pi H}{n\mathcal{L}} \tag{25}$$

becomes a function of time. Hence equation (21) is a differential equation for V with time dependent coefficients. As will be discussed further on, appropriate constant values of $\mathcal{L}$ and H may be chosen as a simplifying approximation.

Internal Buckling under Rigid Confinement

By assuming the center lines of two competent layers A and B to remain straight and to be separated by one-half wave length H, we obtain the case shown in Figure 4b. It represents a folding confined between two rigid straight boundaries without friction. The two competent layers adjacent to the rigid walls are only half the thickness of the same layers inside the medium. The distance H between the confining walls increases with time to allow for lateral expansion as the medium is compressed. The result is obviously valid for any type of confinement which is approximated by the present case. Frictionless boundaries are not essential since the adjacent incompetent layer tends to act as a lubricant.

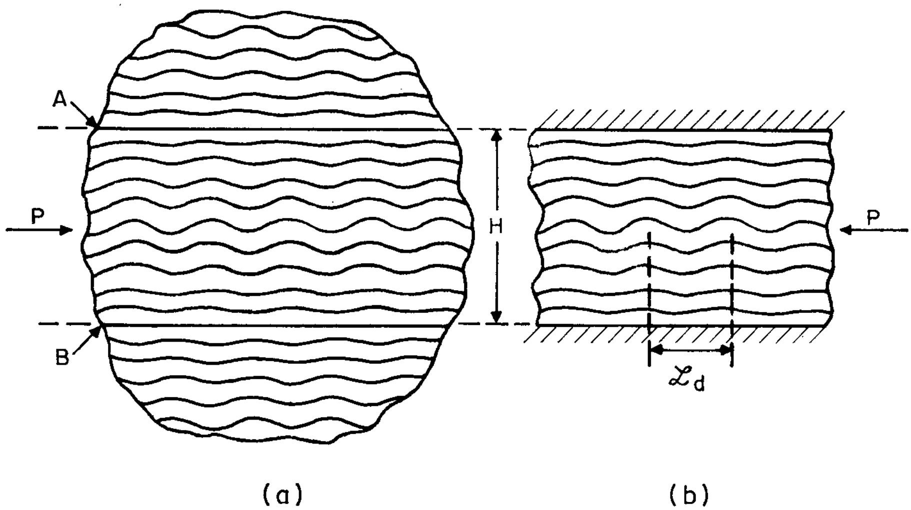

Figure 4. Folding pattern of internal buckling of a multilayered structure. (a), self-confinement; (b), rigid confinement

Layers of Equal Thickness

For competent and incompetent layers of equal thickness equation (21) leads to the result already obtained for this case in a previous paper (Biot, 1964a). We put $\alpha_1 = \alpha_2 = 1/2$. With P_1 denoting the compressive stress in the competent layer, we write approximately $P \cong (1/2)P_1$, $\eta_t \cong 2\eta_2$, $\eta_n \cong (1/2)\eta_1$. With these values equation (21) becomes

$$\frac{P_1}{4\eta_1}\frac{V}{\dot{V}} = \frac{\eta_2}{\eta_1} + \left(\frac{\pi}{2n}\right)^2 \frac{\delta}{\gamma^2} + \tfrac{1}{3}\gamma^2, \tag{26}$$

where

$$\frac{1}{\delta} = 1 + \tfrac{1}{3}\frac{\eta_1}{\eta_2}\gamma^2 . \tag{27}$$

This result coincides with equation (18) of the previous paper (Biot, 1964a).

Dominant Wave length

If the amount of compression is not large, the quantities $\mathcal{L}$ and H may be approximated by time-independent values. In that case $\dot{V}/V = p$ is a constant determined by substituting this value in equation (21), whereas V is proportional to exp (pt).

The right side of equation (21), considered as a function γ goes through a minimum for a certain value $\gamma = \gamma_d$ which yields the dominant wave length $\mathcal{L}_d = \pi h/\gamma_d$. This is the wave length which grows at the fastest rate and will predominate in the folding process. As shown elsewhere (Biot, 1964b; 1965b; 1965c) the time-independent values of $\mathcal{L}$ and H to be used in the simplified approximate treatment are the wave length after deformation and the thickness before deformation, respectively, The evaluation of the dominant wave length is considerably simplified by writing γ^2 in the form

$$\gamma^2 = \frac{\pi\sqrt{3}}{4}\frac{Z^2}{\alpha_1 n}, \tag{28}$$

with a new variable Z related to the wave length by the relation

$$\mathcal{L} = \left(\frac{2\pi}{\sqrt{3}}\right)^{\frac{1}{2}} \frac{\sqrt{h_1 H}}{Z} = 1.90\,\frac{\sqrt{h_1 H}}{Z}. \tag{29}$$

By substituting the value obtained in equation (28) for γ^2 and introducing the approximate relation[2] $\eta_n = \alpha_1\eta_1$, equation (21) is written

$$P\frac{V}{\dot{V}} = \eta_t + \frac{4\pi\alpha_1^2\eta_1}{\sqrt{3}\,n}\left[Z^2 + \frac{\delta}{Z^2}\right]. \tag{30}$$

The factor δ of equations (22) now becomes

$$\delta = \frac{1}{1 + KZ^2}, \tag{31}$$

with

$$K = \frac{4\pi}{\sqrt{3}}\frac{\eta_1\alpha_2^3}{\eta_2 n}. \tag{32}$$

Note the physical significance of δ. As shown previously (Biot, 1964a) it measures an apparent vertical stiffness $\delta\eta_n$ which is smaller than η_n due to interstitial flow.

The dominant wave length is determined by equating to zero the derivative with respect to Z of the bracket in equation (30). This yields a relationship between Z and K which may be written in the form

$$Z^4 = \delta(2 - \delta). \tag{33}$$

For numerical evaluation we substitute this value of Z in expression (31) of δ and obtain

$$K^2 = \frac{(1 - \delta)^2}{\delta^3(2 - \delta)}. \tag{34}$$

Equations (33) and (34) provide a parametric plot of Z versus K, shown in Figure 5, with the values of δ along the curve. The plot may be approximated by two asymptotic branches, AB and BC. Along AB ($Z = 1$) the dominant wave length is

$$\mathcal{L}_d = 1.90\sqrt{h_1 H}. \tag{35}$$

For small values of δ we write $Z^4 = 2\delta$, $K^2 = (1/2)\ \delta^{-3}$, and $Z = (2/K)^{\frac{1}{6}}$. This yields the branch BC and the corresponding value

$$\mathcal{L}_d = 2.36\left(\frac{\eta_1}{\eta_2}\frac{\alpha_2^3}{\eta}\right)^{\frac{1}{6}}\sqrt{h_1 H}. \tag{36}$$

Hence the dominant wave length is given by whichever is the largest of the two values, equation (35) or (36). Note that the same approximations are also derived by substituting $\delta = 1$ and $\delta = 1/\kappa\gamma^2$ in equation (21). This remarkably simple result shows that the wave length is controlled mainly by the thickness h_1 of the competent layer and by the confining distance H.

The effect of interstitial flow on $\mathcal{L}_d$ becomes significant for $K > 2$ when the value (36) is applicable. This is expressed by the condition

$$\frac{\eta_1}{\eta_2} \gtrsim 0.275\,\frac{n}{\alpha_2^3}. \tag{37}$$

For layers of equal thickness ($\alpha_2 = 1/2$) we find $(0.275)/\alpha_2^3 = 2.2$. In a previous discussion of this case (Biot, 1964a) the factor 3.4 was used instead of 2.2. The latter is more correct, but the difference is not significant.

Consider for example the case $\eta_1/\eta_2 = 1000$, $n = 100$, and $\alpha_2 = 1/4$. The incompetent layer is one-third the thickness of the competent layer. Application to this case of the inequality (37) shows that interstitial flow is not significant and the dominant wave length is given approximately by the simple formula (35). Note that in equation (36) the correction factor for interstitial flow is very insensitive to the vis-

[2] The analysis may be carried out without this assumption, but in practice little is gained by doing so.

cosity ratio η_1/η_2. This conclusion was derived previously (Biot, 1964a).

Empirical Formula for the Dominant Wave Length

With an error less than about two per cent it is possible to fit the curve of Figure 5 in the complete range ($0 < K < \infty$) by the equation

$$\frac{1}{Z} = \left(1 + \frac{K}{2}\right)^{\frac{1}{6}}. \qquad (38)$$

Hence from expressions (29) and (32) we derive the following empirical formula for the dominant wave length:

$$\mathfrak{L}_d = 1.90\left(1 + 3.63\,\frac{\eta_1 a_2^3}{\eta_2 n}\right)^{\frac{1}{6}}\sqrt{h_1 H}\,. \qquad (39)$$

Range of Validity

If we put $a_2 = 0$ in equation (39) the dominant wave length is given by expression (35). This corresponds to incompetent layers of zero thickness. However, this has a physical meaning only if sufficient lubrication is retained at the interface. As can be seen from the value of equation (7) of η_t interfacial lubrication requires that the value η_2/a_2 remain sufficiently small in relation to η_1.

Owing to the approximations, strict applicability of the theory requires the wave length to be larger than about 10 times the thickness of any layer (Biot, 1964a; 1965c). This will be verified in a very large category of problems.

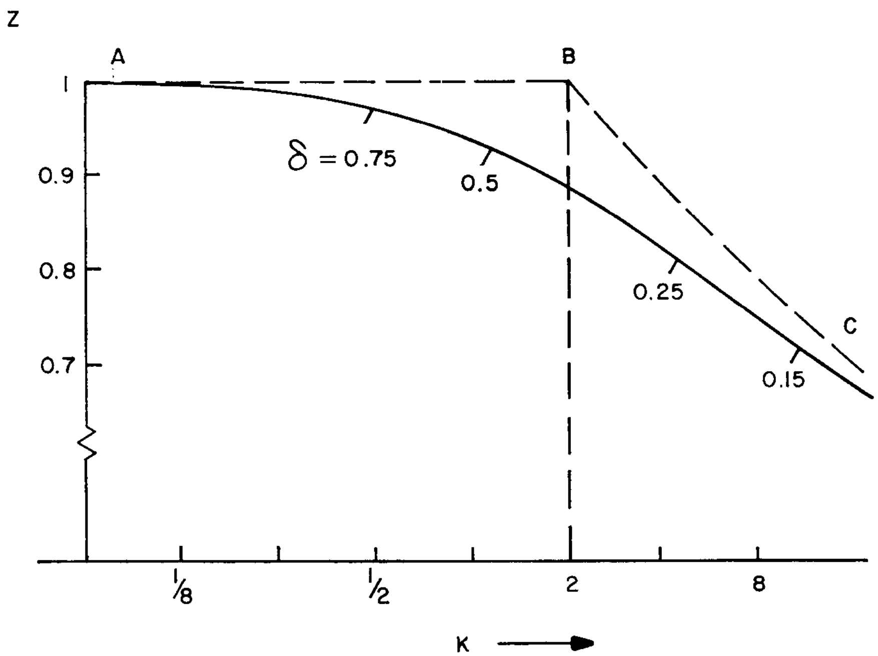

Figure 5. Plot representing in compact form the relation between the dominant wave length and the controlling parameters according to equation (33) for the nondimensional variables Z (29) and K (32). A logarithmic scale is used for K. The relative magnitude of interstitial flow is measured by the value of $1/\delta$.

Large Deformations and Viscous-Fluid Theory

The previous results are applicable to a category of large deformations. As already stated, equation (21) may be considered as a differential equation for V, with time-dependent coefficients. In this case γ is a function of time which embodies the gradual thickening of the layers and the corresponding shortening of the wave length. The theory is therefore applicable to large compressive strain, with moderate slopes. The author has shown that in first approximation the *observed* folding wave length does not participate in the over-all shortening.

More comprehensive treatment for the case of large strain has been given elsewhere (Biot, 1964b; 1965b; 1965c), in the context of viscous-fluid theory. In this connection it is of interest to show that the same equation (9) may be derived directly from viscous-fluid theory. We have shown (Biot, 1964b; 1965b; 1965c) that viscous buckling may be derived from an analog model initially stress-free, where the effect of the compression is replaced by tangential forces at the interface of the two fluids. This amounts to putting $P = 0$ in equation (2) and including in the value of m an additional moment $h_1(P_1 - P_2)\partial v/\partial x$, where P_1 and P_2 are the compressive stresses in the competent and incompetent layers, respectively. This procedure leads to the same equation (9) except that $2Ph$ is replaced by $h_1(P_1 - P_2)$. The difference amounts to a small correction in the time scale which in practice may be disregarded.

Elastic and Viscoelastic Materials

By the correspondence principle the theory is formally the same for viscous, elastic, and viscoelastic media. In particular the buckling load for a perfectly elastic medium is derived by replacing the time derivative $\dot{V}$ by V and the viscosities η_1 and η_2 by the elastic moduli μ_1 and μ_2 of the two layers. This leads to equation (16) of a previous paper (Biot, 1964a). This same equation also provides the solution of a large category of viscoelastic cases. For example, putting $M = \mu_1\alpha_1 + p\eta_2\alpha_2$ and $L = p\mu_1\eta_2/(\alpha_1\eta_2 p + \alpha_2\mu_1)$ in the cited equation solves the folding problem of elastic layers of modulus μ_1 alternating with viscous layers of viscosity η_2. The quantity p appears in the factor exp (pt) which describes the time rate of growth of the amplitude of folding. A detailed discussion of the general theory is given in the author's book (Biot, 1965b).

References Cited

Biot, M. A., 1959, On the instability and folding deformation of a layered viscoelastic medium in compression: Jour. Applied Mechanics, ser. E. v, 26, p. 393–400

—— 1961, Theory of folding of stratified viscoelastic media and its implications in tectonics and orogenesis: Geol. Soc. America Bull., v. 72, p. 1595–1620

—— 1963, Internal buckling under initial stress in finite elasticity: Royal Soc. London Proc., ser. A, v. 273, p. 306–328

—— 1964a, Theory of internal buckling of a confined multilayered structure: Geol. Soc. America Bull., v. 75, p. 563–568

—— 1964b, Theory of viscous buckling of multilayered fluids undergoing finite strain: The Physics of Fluids, v. 7, no. 6, p. 855–859

—— 1965a, Theory of similar folding of the first and second kind: Geol. Soc. America Bull., v. 76, p. 251–258

—— 1965b, Mechanics of incremental deformations: New York, John Wiley and Sons, Inc., 504 p.

—— 1965c, Theory of viscous buckling and gravity instability of multilayers with large deformation: Geol. Soc. America Bull., v. 76, p. 371–378

Biot, M. A., Odé, H., and Roever, W. L., 1961, Experimental verification of the theory of folding of stratified viscoelastic media: Geol. Soc. America Bull., v. 72, p. 1621–1632

Shell Development Company, Houston, Texas
Manuscript Received by the Society April 7, 1965