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THEORY OF STABILITY OF PERIODIC MULTILAYERED CONTINUA

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SUMMARY

The writer's theory of stability of multilayered continua is applied to the case of a periodic alternation of layers of two rubber-like materials of different rigidity in a state of homogeneous finite strain. An exact buckling condition is derived which is remarkably simple. It is solved numerically for the case of layers of equal thickness and the buckling load is plotted as a function of the wavelength for various ratios of the two rigidities. At small wavelength the buckling degenerates into interfacial instability. At large wavelength the instability coincides with internal buckling of an anisotropic continuum equivalent to the multilayered system. The case of layers of different thickness is discussed. By viscoelastic correspondence the results are immediately applicable to viscoelastic media. The general theory also provides a similar solution for layers of anisotropic materials.

1. Derivation of the stability equation

We shall first consider an elastic medium of adhering layers of the same thickness with alternate elastic rigidity (Fig. 1). The medium is assumed incompressible with isotropic finite stress-strain relations corresponding to a rubberlike medium as derived by Treloar (**1**). The principal extension ratios in the initial state of stress are denoted by $\lambda_1,\lambda_2,\lambda_3$. The condition of incompressibility is

$$\lambda_1\lambda_2\lambda_3 = 1 \tag{1}$$

The extension λ_1 is parallel with the layers while λ_2 is oriented in the perpendicular direction. The principal stresses S_{11} S_{22} S_{33} associated with this strain satisfy the relations, (**2**),

$$\begin{aligned} S_{22}-S_{33} &= \mu_0(\lambda_2^2-\lambda_3^2) \\ S_{33}-S_{11} &= \mu_0(\lambda_3^2-\lambda_1^2) \\ S_{11}-S_{22} &= \mu_0(\lambda_1^2-\lambda_2^2) \end{aligned} \tag{2}$$

The coefficient μ_0 is the shear modulus of one layer in the stress free state. In the other layer of shear modulus μ_{01} the stresses are $S_{11}^{(1)}\,S_{22}^{(1)}\,S_{33}^{(1)}$. They satisfy relations (2) with the coefficient μ_{01} replacing μ_0. The last equation (2) for this layer is

$$S_{11}^{(1)}-S_{22}^{(1)} = \mu_{01}(\lambda_1^2-\lambda_2^2) \tag{3}$$

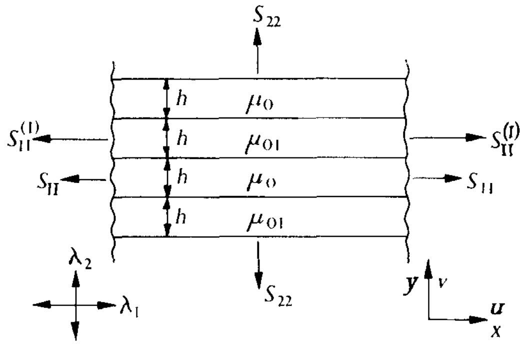

FIG. 1. Elastic layers of equal thickness h and alternate rigidities μ_0 and μ_{01} under initial stress. The number of layers is infinite.

Consider a state of plane strain superposed upon this initial finite strain. The incremental deformation is in the plane of Fig. 1, and, referred to x, y axes, parallel with the finite extensions λ_1 λ_2.

We are concerned with the problem of stability of this periodic-layered system and particularly with a mode of buckling which has the same periodicity as the layers. This means that the deformation of each layer is antisymmetric with respect to its middle plane (Fig. 2). The deformation is assumed sinusoidal along x. Consider an interface with the layer μ_{01} on top and the layer μ_0 on the bottom. The incremental displacement of the interface is written

$$\begin{aligned} u &= U \sin lx \\ v &= V \cos lx. \end{aligned} \tag{4}$$

The incremental stress acting at the interface on the bottom layer is written

$$\begin{aligned} \Delta' f_x &= \tau \sin lx \\ \Delta' f_y &= q \cos lx. \end{aligned} \tag{5}$$

These stresses are normal and tangential components to the deformed surface. We have shown (**3, 4**) that the displacements and the stresses are related by the equations

$$\begin{aligned} \frac{\tau}{lL} &= a_{11}U + a_{12}V \\ \frac{q}{lL} &= a_{12}U + a_{22}V \end{aligned} \tag{6}$$

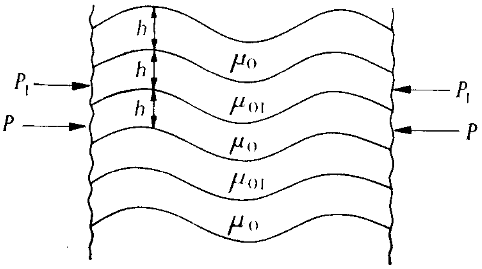

FIG. 2. Buckling mode of the infinite multilayered system. The 'effective' compressions in each layer at buckling are $P = S_{22} - S_{11}$ and $P_1 = S_{22} - S_{11}^{(1)}$.

The coefficients of these equations are

$$a_{11} = \frac{1-k^2}{z_1 - z_2}$$

$$a_{12} = \frac{2z_2 - (1+k^2)z_1}{z_1 - z_2} \tag{7}$$

$$a_{22} = a_{11} z_1 z_2.$$

with

$$L = \tfrac{1}{2}\mu_0(\lambda_1^2 + \lambda_2^2)(1+\zeta)$$

$$z_1 = \tanh \gamma$$

$$z_2 = k \tanh k\gamma$$

$$\gamma = \tfrac{1}{2} lh \tag{8}$$

$$k = \sqrt{\frac{1-\zeta}{1+\zeta}}$$

$$\zeta = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2 + \lambda_1^2}$$

Consider now the layer of modulus μ_{01}. The same relation as (6) may be written for the normal and tangential increments of the stresses in that layer. The parameters γ, k, and ζ are the same in both layers. The only difference appears in the value of L which becomes†

$$L' = \tfrac{1}{2}\mu_{01}(\lambda_1^2 + \lambda_2^2)(1+\zeta) \tag{9}$$

† The simpler values may also be written,

$$k = \lambda_1/\lambda_2 \quad L = \mu_0 \lambda_2^2 \quad L' = \mu_{01}\lambda_2^2$$

However, it is of interest to keep the parameter ζ in view of its significance for viscoelasticity and more general problems.

With this value L^1 the stresses in this layer are given by the equations

$$\frac{\tau}{lL'} = -a_{11}U + a_{12}V$$
$$\frac{q}{lL'} = a_{12}U - a_{22}V. \tag{10}$$

The coefficients a_{11} and a_{22} are preceded by a minus sign because the stresses and displacements are those of the lower face of the layer. Since perfect adherence is assumed and because the stresses are continuous at the interface, we may equate the values of τ and q given by equations (9) and (10). This yields

$$\begin{aligned} L(a_{11}U + a_{12}V) &= L^1(-a_{11}U + a_{12}V) \\ L(a_{12}U + a_{22}V) &= L^1(a_{12}U - a_{22}V) \end{aligned} \tag{11}$$

Equating to zero the determinant of this homogeneous system we write

$$\left(\frac{1-n}{1+n}\right)^2 = \frac{a_{11}a_{22}}{a_{12}^2} \tag{12}$$

The rigidity ratio is

$$n = \frac{\mu_{01}}{\mu_0} \tag{13}$$

The characteristic equation (12) represents the buckling condition of the multilayered system for the particular periodic mode of instability where all vertical interfacial displacements are the same. Attention is called to the physical significance of the variable ζ. It may be written

$$\zeta = \frac{P}{2\mu} = \frac{P_1}{2\mu_1} \tag{14}$$

The quantities,

$$\mu = \tfrac{1}{2}\mu_0(\lambda_1^2 + \lambda_2^2)$$
$$\mu_1 = \tfrac{1}{2}\mu_{01}(\lambda_1^2 + \lambda_2^2) \tag{15}$$

represent the incremental shear moduli for the two layers under initial stress. The values of P and P_1 are

$$P = \mu_0(\lambda_2^2 - \lambda_1^2) = S_{22} - S_{11}$$
$$P_1 = \mu_{01}(\lambda_2^2 - \lambda_1^2) = S_{22} - S_{11}^{(1)} \tag{16}$$

They represent the 'effective' compressive stress acting in each layer along its axis. Before solving the characteristic equation (12) it is useful to establish the nature of its numerical solution by considering the two limiting cases $\gamma = 0$ and $\gamma = \infty$.

2. Limiting case of infinite wavelength—Internal buckling

For large wave lengths the value of γ tends to zero. For small values of γ we write the limiting approximations

$$\begin{aligned} z_1 &= \gamma \\ z_2 &= k^2\gamma. \end{aligned} \tag{17}$$

With the values (17) the buckling condition (12) becomes

$$\left(\frac{1-n}{1+n}\right)^2 = k^2 \tag{18}$$

or

$$\frac{2}{\zeta} = n+\frac{1}{n} \tag{19}$$

It is easy to interpret this result noting that for large wavelengths or, what is the same thing, for very thin layers, the medium behaves as an anisotropic continuum. It was shown (**5**) that the slide modulus L_{av} for this equivalent continuum is

$$L_{av} = \frac{1}{\dfrac{1}{2L}+\dfrac{1}{2L'}} \tag{20}$$

From equations (8), (9), and (15) we derive

$$\begin{aligned} L &= \mu(1+\zeta) \\ L^1 &= \mu_1(1+\zeta) \end{aligned} \tag{21}$$

Hence

$$L_{av} = (1+\zeta)\frac{2\mu\mu_1}{\mu+\mu_1} \tag{22}$$

The average effective compressive stress in the equivalent continuum is

$$P_{av} = \tfrac{1}{2}(P+P_1) \tag{23}$$

By taking into account relations (14), we write

$$P_{av} = \zeta(\mu+\mu_1). \tag{24}$$

It was shown (**5**) that the condition of internal buckling of such an anisotropic continuum is

$$L_{av} = P_{av} \tag{25}$$

Substituting the values (22) and (24), this equation reduces to

$$\frac{2}{\zeta} = n+\frac{1}{n} \tag{26}$$

which is identical with the limiting equation (19). Hence at large wavelengths the instability coincides with the phenomenon of internal buckling of an anisotropic continuum.

3. Limiting case of small wavelength—Interfacial instability

For small wavelength we put $\gamma = \infty$. The values of z_1 and z_2 become

$$\begin{aligned} z_1 &= 1 \\ z_2 &= k \end{aligned} \tag{27}$$

Substituting these values in the characteristic equation (12), we find

$$\left(\frac{1-n}{1+n}\right)^2 = \left(\frac{1+k}{1-k}\right)^2 k \tag{28}$$

This equation coincides with the condition of interfacial instability of two adhering infinite half spaces analysed in a previous paper (**6**).

4. Numerical solution of the stability equation

Equation (12) has been solved numerically and the stability parameter ζ has been plotted as a function of the wavelength variable γ for a number of values of the rigidity ratio n. Results are shown in Fig. 3. We note that $1/\gamma$ is proportional to the ratio of the wavelength to the layer thickness. For $\gamma = 0$ the values of ζ are given by equation (19) corresponding to internal buckling of the equivalent continuum. As the

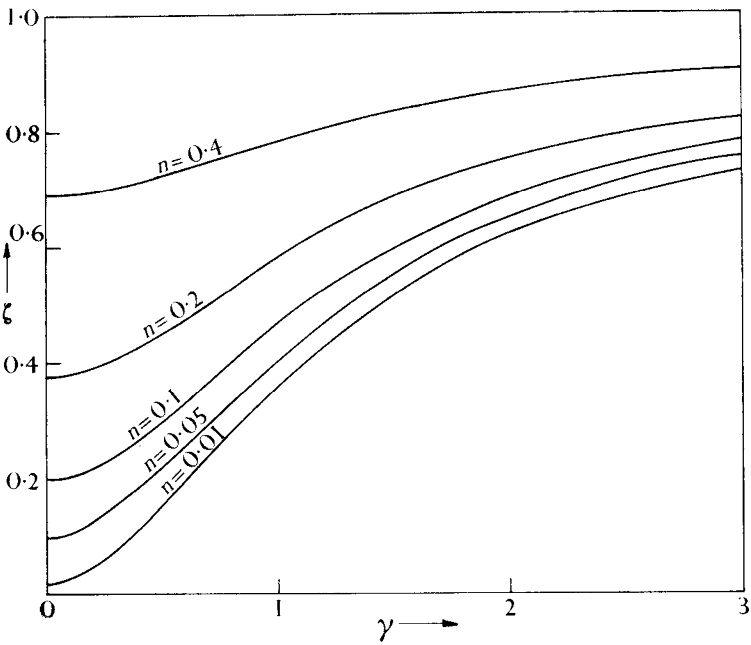

FIG. 3. Stability parameter ζ versus the wavelength parameter γ for different values of the rigidity ratio $n = \mu_{01}/\mu_0$.

wavelength decreases the bending stiffness of the layers enters into play and the value of ζ increases. For $\gamma = \infty$ i.e. for very short wavelengths the curves for ζ tend toward horizontal asymptotes corresponding to interfacial instability, and determined by equation (28).

5. Viscoelastic correspondence

The present solution is applicable to a viscoelastic multilayered medium provided we replace the elastic incremental coefficients by operators. For example, if incremental deformations are purely viscous we substitute the operators

$$\begin{aligned} \bar{\mu} &= \eta p \\ \bar{\mu}_1 &= \eta_1 p \end{aligned} \tag{29}$$

in place of the elastic coefficients μ and μ_1. The operator $p = d/dt$ is the time differential while η and η_1 are viscosity coefficients. The parameter n becomes the viscosity ratio

$$n = \frac{\eta_1}{\eta} \tag{30}$$

The parameter ζ is

$$\zeta = \frac{P}{2\eta p} = \frac{P_1}{2\eta_1 p} \tag{31}$$

This shows that in each layer the initial compressive stress must be assumed proportional to the respective viscosity coefficients. For a given viscosity ratio and a given wavelength and amplitude of folding grows at an exponential rate proportional to $\exp(pt)$ whose p is determined from the value of ζ by using the same diagram as in Fig. 3.

Strictly speaking, the correspondence is valid only if the medium is at rest under the initial stress. However, as shown in earlier discussions, for all practical purposes the results are valid for a medium with initial strain rate.

We are assuming here that the instability is in the nature of a creep buckling where inertia effects are negligible. A more complete theory including inertia forces and compressibility has been developed in another paper (**7**).

6. Extension to layers of different thickness and anisotropic properties

The present results are easily extended to the case of a multilayered system of layers of alternating thickness h and h_1. Equations (11) are replaced by

$$\begin{aligned} L(a_{11}U + a_{12}V) &= L'(-a'_{11}U + a'_{12}V) \\ L(a_{12}U + a_{22}V) &= L'(a'_{12}U - a'_{22}V) \end{aligned} \tag{32}$$

The coefficients a'_{ij} now refer to the layer of thickness h_1 and are obtained by putting $\gamma = \frac{1}{2}lh_1$ into the values of z_1 and z_2. For $h_1 = \infty$ the stability is the same as for the single embedded layer analyzed previously (**8**). In this case Fig. 3 is replaced by the diagram of Fig. 2 in the previous paper (**8**). Actually it is not essential that h_1 be infinite for this case. It is sufficient that the wavelength and the thickness h_1 be such that

$$\tanh(\tfrac{1}{2}klh_1) \simeq 1 \tag{33}$$

This will be the case if the wavelength is smaller than about kh_1. This result indicates the general nature of the solution for intermediate values of the thickness ratio h/h_1.

Equations (32) are also valid for anisotropic media. The coefficients for this case were obtained in a previous detailed analysis (**3**).

Finally, the case of viscoelastic layers of alternating thickness and anisotropic properties is treated in the same way by viscoelastic correspondence (**9**).

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