Variational Analysis of Ablation

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THE VARIATIONAL and Lagrangian thermodynamics developed in earlier publications[1, 2] are directly applicable to problems of heat conduction with melting boundaries. These techniques are used here in treating the problem of a half-space subjected to a constant rate of heat input at the melting surface (Fig. 1).

The applicability of the Lagrangian equations to this case follows from the fact that the basic variational principle is valid whether the boundaries are fixed or move as arbitrary functions of time. This can be seen if we remember that the equations govern only the instantaneous configuration of the flow rates for a given geometry and temperature field.

The numerical results obtained in the present analysis are in very satisfactory agreement with the exact solution of Landau.[3] This is in contrast with previous applications of variational methods to this problem.[4]

In Ref. 2 the Lagrangian heat-flow equations are derived in the form

$$(\partial V/\partial q_i) + (\partial D/\partial \dot{q}_i) = Q_i \tag{1}$$

The general expressions for the thermal potential (V), dissipation function (D), and generalized thermal force (Q_i) reduce to the following for a cylinder of unit cross-sectional area in the unmelted solid (Fig. 1):

$$V = \frac{1}{2}\iiint c\theta^2 d\tau = \frac{1}{2}\int_a^{a+q_1} c\theta^2 dy \tag{2}$$

$$D = \frac{1}{2}\iiint \frac{1}{k}\dot{\mathbf{H}}^2 d\tau = \frac{1}{2}\int_a^{a+q_1} \frac{1}{k}\dot{H}_y^2 dy \tag{3}$$

$$Q_i = \iint \theta_a \frac{\partial H_n}{\partial q_i}\, dS = \theta_a \left.\frac{\partial H_y}{\partial q_i}\right|_{y=a} \tag{4}$$

We denote the thermal conductivity by k, the heat capacity per unit volume by c, the temperature rise due to heating by θ, and the applied temperature on the boundary of the cylinder by θ_a.

The results in Fig. 2 were obtained assuming a cubic temperature profile,

$$\theta = \theta_m[1 - [(y - a)/q_1]^3 \tag{5}$$

where θ_m is the melting temperature of the material. In the present application $\theta_a = \theta_m$. The penetration distance, q_1, is the only generalized coordinate used in describing the temperature distribution.

The heat flow in the y-direction per unit area is denoted by H_y. Admissible functions for θ and H_y are subject to the requirement that heat be conserved locally, which was treated as a constraint in developing the variational methods.[1, 2] A heat-flow distribution consistent with the assumed temperature distribution is readily obtained:

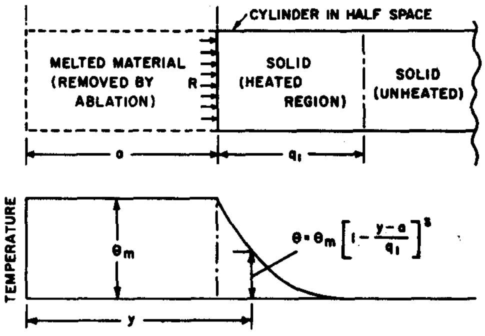

FIG. 1. Ablation of half-space subjected to constant rate of heat input, R, at the melting surface.

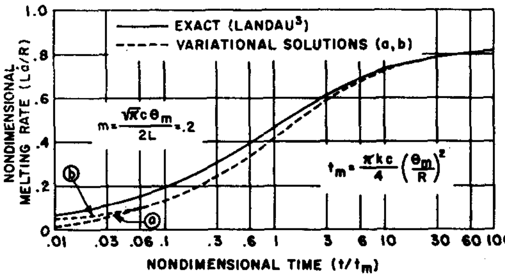

FIG. 2. Comparison between exact and variational solutions for nondimensional melting rate versus nondimensional time.

$$H_y = \int_y^{a+q_1} c\theta dy = \frac{q_1}{4}\theta_m c\left[1 - \frac{y-a}{q_1}\right]^4 \tag{6}$$

Introducing Eqs. (5) and (6) into the preceding equations leads to the following Lagrangian heat-flow equation:

$$q_1[(4/112)\dot{q}_1 + (11/112)\dot{a}] = (5/14)(k/c) \tag{7}$$

This provides a relationship between the velocity of movement of the melting face, $\dot{a}$, and the penetration distance, q_1.

A second equation relating these variables follows from the energy conservation requirement that the total rate of heat input per unit area, R, must equal the sum of the rate of increase in energy of the melted and unmelted regions—i.e.,

$$R = [L + c\theta_m]\dot{a} + \frac{d}{dt}\int_a^{a+q_1} c\theta dy$$

$$= [L + c\theta_m]\dot{a} - {}^1/_4 c\theta_m q_1 \tag{8}$$

The latent heat of melting per unit volume is denoted by L.

Simultaneous solution of Eqs. (7) and (8) gives the variation of nondimensional melting rate ($L\dot{a}/R$) with nondimensional time (t/t_m) presented in Fig. 2. The problem analyzed in this note is limited to the determination of heat conduction after melting starts, since the application of Lagrangian methods to cases without melting has been discussed previously.[2] Time, t, is measured from the start of melting and is made nondimensional by dividing by the time period, t_m, required to bring the solid to the melting temperature with constant heat input ($t_m = \pi k c\theta_m^2/4R^2$).

The penetration distance, q_0, at the start of melting is an initial condition which must be specified. Curve (a) corresponds to a q_0 which will give a zero melting rate ($\dot{a} = 0$) at the start of melting ($t = 0$) as required by the physical nature of the phenomenon. Curve (b) is based on a q_0 found in a Lagrangian analysis of the period prior to melting, assuming a parabolic temperature distribution in this regime.

The sensitivity of the variational solution to the temperature profile assumed in the melting regime was investigated by repeating the calculations, replacing the cubic distribution [Eq. (5)] by a parabolic temperature distribution. The curve for the parabolic distribution corresponding to Curve (a) was slightly below Curve (a) for $t/t_m < 1$ and almost identical with Curve (a) for $t/t_m > 1$. The curve for the parabolic distribution corresponding to Curve (b) is shifted above the exact solution at the start of melting, but is in good agreement with the exact solution for $t/t_m > 0.3$.

The curves in Fig. 2 are computed using the value 0.2 for the parameter $m = \sqrt{\pi}c\theta_m/2L$. Calculations by the variational method were also made for $m = 0$ and showed a satisfactory correlation with the corresponding exact results of Landau, but were less accurate than the solutions obtained at $m = 0.2$. However, the $m = 0$ case is of little practical significance since it cor-

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responds to the unrealistic limiting case of zero melting rate with an infinite heat of liquefaction for $\theta_m > 0$. In the case where m equals zero because the initial temperature of the solid is equal to the melting temperature ($\theta_m = 0$), the melting rate instantaneously reaches the steady-state value, L/R.

References

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